Repeating Decimals

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§1 Fractions to Decimals

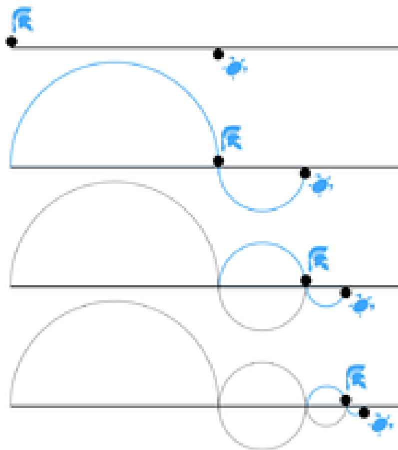

Figure 1: The Story of Achilles and the Turtle: Because Achilles is much faster, he gives the turtle a head start—let's say 10 meters. Achilles starts running, but by the time he reaches the turtle's starting position, the turtle has moved a little farther. When Achilles reaches that new position, the turtle has moved again.This keeps happening infinitely—each time Achilles reaches where the turtle was, the turtle has advanced a bit more.The question is "If Achilles must always reach the tortoise's previous position first, and the tortoise keeps moving forward, does that mean Achilles can never actually catch the tortoise?"

Before we start learning about infinite decimals, let's learn how to write fractions as decimals. Try to convert the following fractions as decimals.

(a)
$$\frac{1}{2} = 0.5$$

(b)
$$\frac{3}{8} = ?$$

(c)
$$\frac{1}{3} = ?$$

(d)
$$\frac{4}{9} = ?$$

§2 Decimals to Fraction

Let's think about the finite decimal 0.375. We can write it as a sum of fractions with denominators that are powers of 10 like this:

$$0.375 = \frac{3}{10} + \frac{7}{100} + \frac{5}{1000}$$

Now consider the following repeating decimal, $0.\bar{4} = 0.444444\ldots$. As a warm-up, try writing it as an infinite sum of fractions whose denominators are powers of 10. What are the first few summands in this infinite summation?

1. (a)
$$0.\bar{4} = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \ldots$$

   (b) What is the value of the infinite sum of fractions you just wrote?

2. a) Write the infinite sum:
$$\frac{15}{100} + \frac{15}{10000} + \frac{15}{1000000} + \ldots$$
   as a repeating decimal and express it as a fraction.

   b) Express the value of their finite sum as a fraction(ratio of integers). (Hint: It's the same as the value of the repeating decimal)

§3 Infinite Geometric Series using Pictures

Now, let's try to evaluate an infinite sum using a different technique. We'll use a geometric picture to help us. The infinite sum is:

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$$

Steps:

(1) Use a blank page for your drawing. Try to draw a good picture; if it's sloppy it will be harder to see what's going on!

(2) Draw a LARGE square and then draw a vertical and horizontal line to divide it into four smaller, equal squares. (Like a window.)

(3) Lightly shade in the lower left small square.

(4) In the upper right small square, again draw a short horizontal and vertical line to divide it into four even smaller squares—let's call these small-small squares.

(5) Lightly shade the lower left small-small square inside the upper right small square.

(6) Repeat the process in the upper right small-small square: Divide it into 4 small-small-small squares, and shade in the lower left of these.

(7) Let's say the side length of the initial square is 1. What does the shaded portion represent?

(8) Why is the total shaded portion equal to $\frac{1}{3}$?

§4 Geometric Series

The square-drawing way of evaluating an infinite sum was kind of cool, but it's not clear how we can use it on other sums we might want to know the value of... So let's try something different this time.

Definition 4.1. A geometric sequence is a sequence where each number of the sequence is a constant times the previous. Let r be the common ratio between the terms.

Consider the sum

$$S = 1 + r + r^2 + r^3 + \dots$$

Also, consider the sum

$$rS = r + r^2 + r^3 + \dots$$

How can we use these two equations to find the value of S? Let's work on the following problem to get a general idea:

Problem 4.2

$$S = \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \dots$$

Hint: Multiply both sides by 2/3.

Theorem 4.3

$$1 + r + r^2 + \cdots = \frac{1}{1-r}, \quad \text{when } -1 < r < 1$$

1. Why is this equation true?
2. Why isnt numbers greater than 1 included? Why isn't 1 and -1 included? Consider the following problem:

   Problem 4.4

   1+2+4+8+...=-1

Problem 4.5

Using this, show that

$$0.999\cdots = 1$$

and similarly

$$0.333\cdots = 1/3$$

Note that this is equal and not approximate.

Problem 4.6

Using a similar idea to the proof of the infinite geometric series, find the value of

$$1 + 2r + 3r^2 + 4r^3 + \dots$$